THE "CRITICAL SURFACES" IN THE PHOTOGRAMMETRIC CARDINAL PROBLEM AND THEIR IMPORTANCE AND ELIMINATION IN PHOTOGRAMMETRIC PRACTICE

by

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This report is a summary of the paper on the "Problem of Critical Surfaces in Theory and Practice" which will be published this year by the German Geodetic Commission. It contains only an exposition of the most important results and omits their rigid development by mathematical calculation.

After previous investigations by E. C. P. Poivilliers, R. Bosshardt and R. Finsterwalder, the first general solution to the problem of "critical surfaces" has been supplied by J. Krames. He found by projective-geometric research that the relative orientation of a picture pair becomes uncertain if

1) the photographed terrain has the form of an orthogonal regular surface of the second order (orthogonal hyperboloid, paraboloid, cone, rotating cylinder, hyperbolic cylinder, pair of planes), and if

2) the photographic base lies on a so called "principal determinant" of the surface; it may be assumed that an orthogonal-regular surface is produced by two coincident congruent bundles of planes supported by the principal determinant.

A discussion as to which of the 8 possible orthogonal regular surfaces of the second order may actually produce a critical case in aero-photogrammetric practice, must start from photographic and morphological possibilities. Critical formations of the terrain may occur in *valleys.* A valley formation of this type must have an adequate length and width so as it can fully cover one pair of aerial photographs. Furthermore, the valley must not open into side valleys within the area, covered by a picture pair.

Otherwise, control points outside the critical area will be available for unequivocal relative orientation. Since a photographic flight is usually made on a level parallel to the bottom of the valley and at a more or less constant height above the bottom, the orthogonal hyperboloid and the orthogonal cone are eliminated right away as critical surfaces for all practical purposes. In either case, the principal determinants are in an oblique position with reference to the valley bottom as regards position and elevation. They may therefore not be used as flight paths or photographic bases.

The principal determinants of the orthogonal paraboloid, which must be on the ground itself, cannot be bases for an air photograph. This also applies to the hyperbolic cylinder and the orthogonal pair of planes. Hence, there remains only the rotating cylinder as a critical surface, provided the base of the photograph is on one of the straight lines of its convex surface. The rotating cylinder on which the base is on the vertex convex surface line has already been recognized by Bosshardt and Finsterwalder to be the most important critical surface. In practice, the hyperboloid and the cone may become critical surfaces if their shape varies but very slightly from that of the rotating cylinder. However, these variations are practically negligible.

At first, these considerations are of a purely projective nature and do not supply any information on the analytical character of the problem of uncertainty in the concrete individual case. Such information may only be derived from an analytical treatment of the critical surfaces which must start from the equation of errors of relative orientation. This analysis shows that the uncertainty consists in a linear interdependence of the 5 orientation elements.

A y-parallax which is caused by errors of one or more orientation elements, may be eliminated on the entire critical surface by modifications of the other orientation elements. On the orthogonal hyperboloid, all five orientation elements are involved in this interdependence. On the rotating cylinder, which is practically the only important case, only three orientation elements are still involved, that is, — starting from the matching of consecutive pictures, — the displacements b_y and b_z of the right camera station and the tilt differential ω. An investigation of the vertical section of the cylinder through the right camera station furnishes a very simple geometrical explanation for this uncertainty and for the interdependence of the three above mentioned orientation elements.

A displacement by db_y and db_z of the right camera station, which returns it from its correct position to the cylinder, may be compensated in all points of the cylinder section and *only* in these points by one and the same $d\omega$. For peripheric angles above the same arc are equal. A relief model free from y-parallaxes can therefore also be obtained in this manner with an orientation which contains errors in db_y, db_z and $d\omega$, for the parallaxes produced in the entire model by an orientation element that is in error, are always eliminated by changes of the other two orientation elements.

However, this model does no longer bear resemblance to the terrain which has been photographed. It contains deformations in which errors of elevation are particularly prominent. Therefore if it should be possible to check the elevations with the aid of surplus control points within the scope of absolute orientation, the observed elevations errors permit to correct relative orientation and thus eliminate the uncertainly of orientation. On general principle, elevation control in *one* additional point will be sufficient,

The ascertained error of elevation permits to compute one of the three orientation errors. Owing to the interdependence of the orientation errors, the other two may be determined without an additional elevation control point. The corresponding formulas are very simple. They may be:

$$d\omega = -\frac{1}{y - 2m} \cdot dh$$

$$db_y = \frac{2n}{y - 2m} \cdot dh$$

$$db_z = \frac{2m}{y - 2m} \cdot dh,$$

where y is the coordinate of the elevation control point in the base system, m and n the horizontal and/or vertical distances of the cylinder axis from the base, and dh the observed elevation error.

The errors of the orientation elements are intrinsically *differential* values which cannot exceed a certain magnitude. A complete compensation of the orientation errors occurs only in the vertical section through the right camera station. In the other cylinder sections, y-parallaxes which are small values of the second order will be present, if the orientation errors are small values of the first order. The condition that this parallax may not exceed the limit, set by the possibility of observation in the plotting instrument, permits to estimate possible maximum of orientation errors. Under normal photographic conditions, the results of this estimation in precision plotting instruments will be:

Standard-angle views	Wide-angle views
$d\omega^{max} = 45.3'$	$d\omega^{max} = 34.4'$
$db_y^{max} = 4.0\,mm$	$db_y^{max} = 3.0\,mm$
$db_z^{max} = 1.0\,mm$	$db_z^{max} = 0.8\,mm$

and for 2nd-order instruments

Standard-angle views	Wide-angle views
$d\omega^{max} = 82.6'$	$d\omega^{max} = 62.7'$
$db_y^{max} = 7.2\,mm$	$db_y^{max} = 5.5\,mm$
$db_z^{max} = 1.8\,mm$	$db_z^{max} = 1.6\,mm$

The corresponding maximum elevation errors which may occur in the model, are as follows:

For first-order plotting instruments

Standard-angle views	Wide-angle views
0.57% of the flying height	0.50% of the flying height

For 2nd-order plotting instruments

1.04% of the flying height	0.90% of the flying height

In the critical photographic case and at a flight altitude of 4,000 m, there may be elevation errors of about 20 m for precision instruments and of about 40 m for second-order instruments. These values show that the correction of relative orientation by additional observed elevation control errors is actually possible.

Under the angle of the overall problem of double-point projection in space the critical surfaces will therefore not produce an uncertainty of orientation. Errors of relative orientation which are caused by critical surfaces may always be eliminated within the scope of absolute orientation.