MEASURING UNCERTAINTY IN SPATIAL FEATURES IN A THREE-DIMENSIONAL GEOGRAPHICAL INFORMATION SYSTEM

Chui Kwan CHEUNG and Wenzhong SHI
Dept. of LSGJ, the Hong Kong Polytechnic University
tracy.cheung@hongkong.com

Keywords: geographical information systems, uncertainty, positional error, numerical integration

ABSTRACT
In this paper, uncertainty of spatial features including linear features, areal features and volumetric features in a three-dimensional (3D) vector-based geographical information system (GIS) is studied. Existing uncertainty models for 3D spatial features are divided into two directions: (a) a confidence region model derived from a statistical approach and (b) a reliability model based on a simulation technique. The confidence region model for a spatial feature could provide an area inside which the 'true' location of the spatial feature is with a predefined probability. In the reliability model, uncertainty was measured by a discrepant area, which is formed by the measured location and the 'true' location of a spatial feature and hence was determined in terms of the measured location and the 'true' location of the spatial feature in a mathematical expression. Based on an assumption of error of the spatial feature, uncertainty of the spatial feature could be simulated repeatedly and the average discrepant area would be obtained. However, it is known that simulation is quite time-consuming and cannot provide a precise solution. Hence, this study further proposes the development an analytical model on the reliability model on spatial features in a 3D GIS. The expected value of the discrepant area of the spatial feature is expressed as a multiple integral by a statistical approach. The authors proposed to solve the multiple integral based on a numerical integration, resulting in an approximate solution of the expected discrepant area. This uncertainty model is also compared with an earlier simulation model.

1. INTRODUCTION

A GIS is defined as a software package, which provides users with a tool to input, store, analyze, retrieve and transform geographical data (Cassettari 1993). It is now widely applied in many different areas including military applications, environmental studies and geological exploration. However, geographical data in GIS is not error-free (Heuvelink 1998). The market of GIS will be affected by evaluating uncertainty in GIS to a certain extent.

Uncertainty in GIS may arise from data collection and input in the first step to spatial analyses. Hence, there are many different types of uncertainty in GIS (Burrough and McDonnell 1998). It is virtually impossible to represent the world completely due to the complexity of the geographical world. Some of the man-made utilities such as water pipes and road networks can be represented by points, lines and polygons while most natural phenomena cannot (Burrough 1986). Differences between the database contents and the phenomena they represent are mainly due to the characteristic of phenomena. In addition, measurement errors are introduced during data collection and input and propagated through GIS operations.

There are different approaches to describe uncertainty of linear features in two-dimensional GIS (Caspary and Scheuring 1992; Dutton 1992; Stanfel and Stanfel 1993,1994; Shi 1994; Easa 1995; Shi and Liu 2000). However, little research exists in the modeling of uncertainty in higher dimensional spatial features. Shi (1997, 1998) derived a confidence region model for 3D and N-dimensional linear features from strictly statistical approaches. Later on, the reliability of 3D spatial features, including linear features, areal features and volumetric features were studied by a simulation technique (Shi and Cheung 1999).

Shi and Cheung (1999) earlier assessed the reliability of a 3D spatial feature by calculating the discrepant area between the measured location and the 'true' location of this spatial feature. It was first assumed that nodal error was normally distributed. In such simulation, the measured nodes of the spatial feature were generated and the discrepant area was calculated. After the simulation was repeated many times, the expected value and the variance of the discrepant area for the linear feature (or the discrepancy volume for the area or volumetric feature) were calculated. On the other hand, Stanfel (1996) suggested that a stochastic method could be used to calculate the expected discrepant area in 2D GIS, mainly due to weakness of the simulation technique such as time-consuming and unstable results.

An analytical expression for the expected discrepant area (or volume) is derived mathematically in this study. Theoretically, its exact solution will be obtained automatically in GIS given the measured location and the 'true' location of the spatial feature. As a result GIS users will aware of the uncertainty of the spatial feature from this indicator. However the analytical expression will be expressed in terms of a multiple integral based on a probability theory. This multiple integral is unable to be solved analytically. This paper thus presents an analytical method with a numerical solution to describe uncertainty of three-dimensional spatial features in GIS.

In this paper, we focus on uncertainty model for a 3D spatial feature in a vector-based GIS. Uncertainty of this spatial feature will be determined by discrepancy between the measured location and the 'true' location of the spatial feature. In Session 2, we will explain the discrepancy of spatial features including linear features, areal features and volumetric features. Due to the weakness of simulation technique (as stated above), we will withdraw this technique in this paper and express the expected discrepant value analytically and this mathematical expression is also given in Session 2. A numerical integration given in Session 3 will be implemented in order to find the approximate solution for the expected discrepant value. Finally, some experimental studies will be conducted and their numerical results will be compared with the simulation result from the previous simulation model.

2. DISCREPANCY OF 3D SPATIAL FEATURES

Uncertainty of spatial features is measured by a discrepancy, which is the difference between the measured location and the 'true' location of spatial features. From a statistical point of view, this indicates that a mean of any variable X is close to its actual value. Thus, in this study, the 'true' locations of the nodes of spatial features refer to their mean locations.

For this study, the following two assumptions are made. First, positional error of a node is within an error ellipsoid whose center corresponds to the 'true' location (Stanfel and Stanfel 1994; Easa 1995). Second, the positional error of the nodes is assumed to follow a normal distribution inherent to specific measurement technologies (Stanfel and Stanfel 1993). The authors will therefore model the positional error of spatial features based on positional nodal error of the spatial features.

2.1 DISCREPANCY OF A LINEAR FEATURE

A line segment has two nodes. Its discrepancy is defined as the area whose boundaries are the measured linear feature and the 'true' linear feature. This area is shaded in Fig. 1.

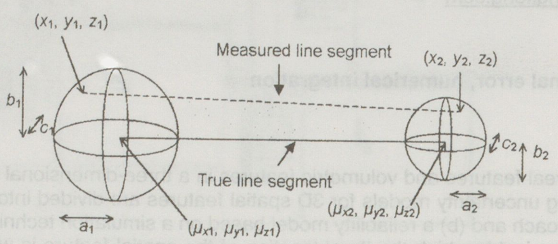

Figure 1. The discrepancy of a line segment

In Fig. 1, the solid line segment is the 'true' line segment joining the 'true' locations of the two nodes (μ_{x1}, μ_{y1}, μ_{z1}) and (μ_{x2}, μ_{y2}, μ_{z2}). The dashed line segment is the measured line segment linking the two measured nodes ($x1$, $y1$, $z1$) and ($x2$, $y2$, $z2$). a_1, b_1 and c_1 are parameters in Eq. 1, which is a mathematical expression of the error ellipsoid for the node on the left-hand side. Similarly, a_2, b_2 and c_2 are parameters in Eq. 2, which is a mathematical expression of the error ellipsoid for the node on the right-hand side. The error ellipsoid is a confidence region for the expected location of the node with a confidence coefficient of (1-α) where $0< \alpha < 1$.

Error ellipsoid for node on the left-hand side

$$
\begin{aligned}
1 = & \left(\frac{x1-\mu_{x1}}{a_1}\right)^2 + \left(\frac{y1-\mu_{y1}}{b_1}\right)^2 + \left(\frac{z1-\mu_{z1}}{c_1}\right)^2 \\
& + 2\left[\frac{\rho_{x1z1}\rho_{y1z1}-\rho_{x1y1}}{k}\right]\left(\frac{x1-\mu_{x1}}{s_{x1}}\right)\left(\frac{y1-\mu_{y1}}{s_{y1}}\right) \\
& + 2\left[\frac{\rho_{x1y1}\rho_{y1z1}-\rho_{x1z1}}{k}\right]\left(\frac{x1-\mu_{x1}}{s_{x1}}\right)\left(\frac{z1-\mu_{z1}}{s_{z1}}\right) \\
& + 2\left[\frac{\rho_{x1y1}\rho_{x1z1}-\rho_{y1z1}}{k}\right]\left(\frac{y1-\mu_{y1}}{s_{y1}}\right)\left(\frac{z1-\mu_{z1}}{s_{z1}}\right)
\end{aligned} \tag{1}
$$

where $k = (-2 \ln(\alpha)) \times (1 - \rho_{x1y1}^2 - \rho_{x1z1}^2 - \rho_{y1z1}^2 + 2\rho_{x1y1}\rho_{x1z1}\rho_{y1z1})$. ρ_{x1y1} is the correlation coefficient of $x1$'s error and $y1$'s error. Similarly, ρ_{y1z1} and ρ_{x1z1} are the correlation coefficients of $y1$'s error and $z1$'s error, and $x1$'s error and $z1$'s error respectively; and s_{x1}, s_{y1} and s_{z1} are standard derivations of $x1$'s, $y1$'s and $z1$'s errors.

Error ellipsoid for node on the right-hand side

$$
\begin{aligned}
1 = & \left(\frac{x2-\mu_{x2}}{a_2}\right)^2 + \left(\frac{y2-\mu_{y2}}{b_2}\right)^2 + \left(\frac{z2-\mu_{z2}}{c_2}\right)^2 \\
& + 2\left[\frac{\rho_{x2z2}\rho_{y2z2}-\rho_{x2y2}}{k}\right]\left(\frac{x2-\mu_{x2}}{s_{x2}}\right)\left(\frac{y2-\mu_{y2}}{s_{y2}}\right) \\
& + 2\left[\frac{\rho_{x2y2}\rho_{y2z2}-\rho_{x2z2}}{k}\right]\left(\frac{x2-\mu_{x2}}{s_{x2}}\right)\left(\frac{z2-\mu_{z2}}{s_{z2}}\right) \\
& + 2\left[\frac{\rho_{x2y2}\rho_{x2z2}-\rho_{y2z2}}{k}\right]\left(\frac{y2-\mu_{y2}}{s_{y2}}\right)\left(\frac{z2-\mu_{z2}}{s_{z2}}\right)
\end{aligned} \tag{2}
$$

where $k = (-2 \ln(\alpha)) \times (1 - \rho_{x2y2}^2 - \rho_{x2z2}^2 - \rho_{y2z2}^2 + 2\rho_{x2y2}\rho_{x2z2}\rho_{y2z2})$. ρ_{x2y2} is the correlation coefficient of $x2$'s error and $y2$'s error. Similarly, ρ_{y2z2} and ρ_{x2z2} are the correlation coefficients of $y2$'s error and $z2$'s error, and $x2$'s error and $z2$'s error respectively; and s_{x2}, s_{y2} and s_{z2} are standard derivations of $x2$'s error, $y2$'s error and $z2$'s error.

The shaded area in Fig. 1 is determined differently depending on its case. Three possible cases exist: (a) the measured locations of all nodes and their corresponding 'true' locations are on a flat plane but the measured and the 'true' line segments do not intersect; (b) the measured and the 'true' line segments intersect; (c) neither case (a) or case (b) is a possibility.

In the first case, it is assumed that the measured locations and the 'true' locations of all nodes should be on a flat plane and that their two corresponding line segments should not intersect. In such a situation, the shaded area can be denoted by area_quad ($x1$, $y1$, $z1$, $x2$, $y2$, $z2$), which is a function of ($x1$, $y1$, $z1$) and ($x2$, $y2$, $z2$), as shown in Eq. 3.

$$
\begin{aligned}
& area\_quad \\
& = 0.5 * (\text{the magnitude of } (A \times B) + \text{the magnitude of } (C \times D))
\end{aligned} \tag{3}
$$

where $A = (x1-\mu_{x1},\ y1-\mu_{y1},\ z1-\mu_{z1})$
$B = (\mu_{x2}-\mu_{x1},\ \mu_{y2}-\mu_{y1},\ \mu_{z2}-\mu_{z1})$
$C = (x1-\mu_{x2},\ y1-\mu_{y2},\ z1-\mu_{z2})$
$D = (x2-\mu_{x2},\ y2-\mu_{y2},\ z2-\mu_{z2})$ and
$A \times B$ is a vector product of A and B and so on.

The second case is illustrated in Fig. 2 whereby the discrepancy is shaded. The shaded area consists of two triangles whose vertices are ($x1$, $y1$, $z1$), (μ_{x1}, μ_{y1}, μ_{z1}) and ($x12$, $y12$, $z12$) for the triangle on the left-hand side, and ($x2$, $y2$, $z2$), (μ_{x2}, μ_{y2}, μ_{z2}) and ($x12$, $y12$, $z12$) for the triangle on the right-hand side. ($x12$, $y12$, $z12$) is an intersecting point on the 'true' line segment and the measured line segment. The discrepancy is denoted by area_triangle which is a function of ($x1$, $y1$, $z1$), ($x2$, $y2$, $z2$), and ($x12$, $y12$, $z12$). This shaded area is expressed in Eq. 4.

$$
\begin{aligned}
& area\_triangle \\
& = 0.5 * (\text{the magnitude of } (A' \times B') + \text{the magnitude of } (C' \times D'))
\end{aligned} \tag{4}
$$

where $A' = (x1-\mu_{x1},\ y1-\mu_{y1},\ z1-\mu_{z1})$
$B' = (x12-\mu_{x1},\ y12-\mu_{y1},\ z12-\mu_{z1})$
$C' = (x12-\mu_{x2},\ y12-\mu_{y2},\ z12-\mu_{z2})$ and
$D' = (x2-\mu_{x2},\ y2-\mu_{y2},\ z2-\mu_{z2})$.

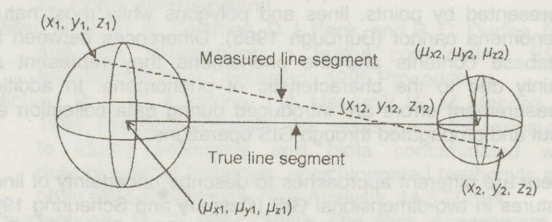

Figure 2. The discrepancy of a line segment forming two triangles

It is also possible that both the measured and the 'true' nodes are neither on a 'flat' plane nor intersect. Under these circumstances the shaded area cannot be obtained exactly. The obscurity of the equation formed by the measured and 'true' nodes affects the discrepancy; the discrepancy cannot be readily calculated. To simplify and quantify such a case, the approximate area of the shaded region in Fig. 1 is described by Eq. 3. Due to errors of the spatial feature's nodes, the expected discrepant area of the line segment is computed as per Eq. 5.

$$
\begin{aligned}
& E(discrepancy) \\
& = \int_{R_1 \cup R_2} f(x1, y1, z1, x2, y2, z2)\, area\_quad\ dz2dy2dx2dz1dy1dx1 \\
& + \int_{R_1 \cup R_2} f(x1, y1, z1, x2, y2, z2)\, area\_triangle\ dz2dy2dx2dz1dy1dx1
\end{aligned} \tag{5}
$$

where R_1 and R_2 are the regions of the two ellipsoids expressed in Eq. 1 and Eq. 2 respectively; U is union of the two regions; and $f(x1,y1,z1,x2,y2,z2)$ is a probability density function (pdf) of normal distribution. The latter's mathematical expression is shown below

$$f(x1,y1,z1,x2,y2,z2) = \frac{1}{(2\pi)^{6/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(x1-\mu_{x1},y1-\mu_{y1},z1-\mu_{z1},x2-\mu_{x2},y2-\mu_{y2},z2-\mu_{z2})\Sigma^{-1}\begin{pmatrix} x1-\mu_{x1} \\ y1-\mu_{y1} \\ z1-\mu_{z1} \\ x2-\mu_{x2} \\ y2-\mu_{y2} \\ z2-\mu_{z2} \end{pmatrix}} \quad (6)$$

where Σ is the covariance matrix of $x1$'s, $y1$'s, $z1$'s, $x2$'s, $y2$'s, and $z2$'s errors.

Joining several line segments together yields a polyline, which is a broad linear feature. The reliability of the linear feature is measured by the discrepancy between the measured location and the ‘true’ location of the linear feature. In this instance, the linear feature only refers to an acyclic polyline.

The discrepancy between the measured location and the ‘true’ location of the linear feature is shown in Fig. 3. The solid linear feature is the ‘true’ location of the linear feature and the dashed linear feature is the measured location. The ‘true’ location is connected by three ‘true’ nodes (μ_{x1}, μ_{y1}, μ_{z1}), (μ_{x2}, μ_{y2}, μ_{z2}) and (μ_{x3}, μ_{y3}, μ_{z3}). The measured location is connected by three measured nodes ($x1$, $y1$, $z1$), ($x2$, $y2$, $z2$) and ($x3$, $y3$, $z3$). The shaded area in Fig. 3 represents the discrepancy between the measured location and the ‘true’ location. As a result, the expected discrepant area is presented in Eq. 7.

$$E(discrepancy) = \int_{R_1 \cup R_2 \cup R_3} f \times area \; dz3dy3dx3\,dz2dy2dx2dz1dy1dx1 \quad (7)$$

where R_1, R_2 and R_3 are regions of the three ellipsoids for three nodes of the linear feature; U is union of regions; f is a pdf of normal distribution; and *area* is the shaded area in Fig. 3.

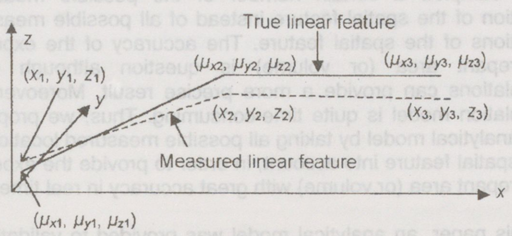

Figure 3. The discrepancy of a linear feature

2.2 DISCREPANCY OF AN AREAL FEATURE

Areal features, as discussed in this paper, refer to polygons in the digital database sense. Though the reliability of an areal feature can be appraised by the discrepancy between the measured location and the ‘true’ location of the areal feature, its definition of discrepancy is distinct from that of linear features. The discrepancy of the areal feature refers to a volume of the area whose boundaries are the measured areal feature and the ‘true’ areal feature. According to the definition for linear features, the discrepancy should be the surface area bounded by the measured linear feature and the ‘true’ linear feature. Fig. 4 and Fig. 5 illustrate this difference in conformity with respect to area and volume.

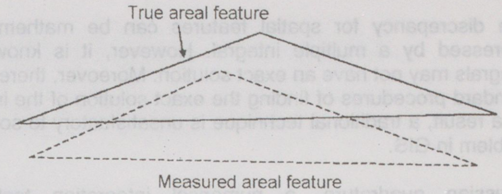

Figure 4. The discrepancy of an areal feature defined by area

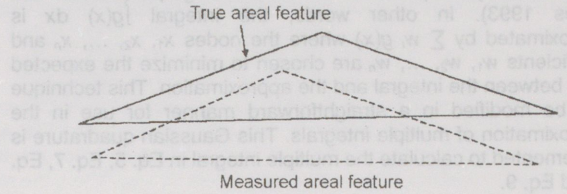

Figure 5. The discrepancy of an areal feature defined by volume

Fig. 4 relates to the discrepancy of the areal feature based on area, and Fig. 5 relates to the discrepancy of the areal feature based on volume. The solid and the dashed areal features are the ‘true’ location and the measured location of the areal feature respectively. Both the area of the shaded area in Fig. 4 and the volume of the shaded area in Fig. 5 represent the discrepancy. Since discrepancy is the difference between reality and users' representation of reality, using the volume of the shaded area to express the discrepancy of areal feature is satisfactory. For instance, the expected discrepant area of an areal feature containing three nodes can be given as follows.

$$E(discrepancy) = \int_{R_1 \cup R_2 \cup R_3} f \times volume \; dz3dy3dx3dz2dy2dx2dz1dy1dx1 \quad (8)$$

where R_1, R_2 and R_3 are regions of the three ellipsoids for three nodes of the areal feature; U is union of regions; f is a pdf of normal distribution; and *volume* is the volume of the shaded object in Fig. 5.

2.3 DISCREPANCY OF A VOLUMETRIC FEATURE

In a 3D GIS, another important element is volumetric features. The difference between the measured location and the ‘true’ location of a volumetric feature is a measure of the reliability of the volumetric feature. This discrepancy can be viewed as a union of surfaces' discrepancies. For example, a volumetric feature contains four nodes and therefore four surfaces. For each surface, its corresponding discrepancy is computed. The discrepancy of the volumetric feature is computed by the union of all of the surfaces' discrepancies. The expected discrepancy is illustrated in Eq. 9 whereby the volumetric feature consists of four nodes.

$$E(discrepancy) = \int_{R_1 \cup R_2 \ldots R_4} f \times volume \, dz4dy4dx4dz3dy3dx3dz2dy2dx2dz1dy1dx1 \quad (9)$$

where R_1, R_2, R_3 and R_4 are regions of the four ellipsoids for four nodes of the volumetric feature; U is union of regions; f is a pdf of normal distribution; and *volume* is the union of the four surfaces' discrepancies.

This expected discrepancy might differ from that in the dependence case. This could indicate that the covariance matrix of f is not diagonal.

3. NUMERICAL INTEGRATION

The discrepancy for spatial features can be mathematically expressed by a multiple integral. However, it is known that integrals may not have an exact solution. Moreover, there are no standard procedures of finding the exact solution of the integral. As a result, a traditional technique is unsatisfactory to solve this problem in GIS.

Gaussian quadrature, a numerical integration technique, approximates the integral of a function by integrating the linear function that joins points of the function's graph (Burden and Faires 1993). In other words, the integral $\int g(x)\ dx$ is approximated by $\sum w_i\ g(x_i)$ where the nodes $x_1, x_2, \ldots, x_n$ and coefficients $w_1, w_2, \ldots, w_n$ are chosen to minimize the expected error between the integral and the approximation. This technique can be modified in a straightforward manner for use in the approximation of multiple integrals. This Gaussian quadrature is implemented to calculate the multiple integral in Eq. 5, Eq. 7, Eq. 8 and Eq. 9.

4. RESULTS AND DISCUSSION

The analytical model for the uncertainty of spatial features is applied to the example data of Shi and Cheung (1999). The two expected nodes of the line segments are (0, 0, 0) and (1000, 0, 0). a_1, b_1, c_1, a_2, b_2 and c_2 are 100ft, 196ft, 148ft, 30ft, 78ft and 90ft respectively. The covariance matrix, in the pdf of the multivariate normal distribution f, is a 6×6 diagonal matrix with nodes $x1$, $y1$, $z1$, $x2$, $y2$ and $z2$ and these nodal errors are assumed to be independent. The confidence coefficient ($1-\alpha$) is 0.95. The expected discrepant area of the line segment is 62145.0ft^2 (as shown in Table 1).

Table 1. The expected discrepant area of spatial features

Spatial feature	The expected discrepancy		Ratio*
	Analytical model	Simulation model	
Line segment	62145.0ft^2	59344.1ft^2	1.047
Linear	85553.6ft^2	89036.1ft^2	0.961
Areal	15209871.4ft^3	18570274.0ft^3	0.819
Volumetric	37688986.2ft^3	44339983.8ft^3	0.849

$$^{*}\ \text{Ratio} = \frac{\text{analytical result}}{\text{simulated result}}$$

For the polyline, the three expected nodes are (0, 0, 0), (500, 500, 707.1) and (1500, 500, 707.1). a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3 and c_3 are 100ft, 196ft, 148ft, 30ft, 78ft, 90ft, 100ft, 196ft and 148ft respectively. The covariance matrix in f is a 9×9 diagonal matrix. The confidence coefficient ($1-\alpha$) is 0.95. The expected discrepant area of the line feature is 85553.6ft^2. Using the three nodes of the polyline for the areal feature results in an expected discrepant area equal to 5209871.4ft^3.

For the example of a volumetric feature, an additional node (to the existing three) is considered. This addition now specifies a volumetric feature. This additional node is (500, 707.1, 500), and its error ellipsoid has parameters a_4 = 30ft, b_4 = 78ft and c_4 = 90ft. The expected discrepant area of the volumetric feature is 37688986.2ft^3.

In Table 1, the expected discrepant areas of the spatial features are recorded for both the numerical integration and the simulation techniques. The ratio from the analytical model to that from the simulation model is in the range of 0.8 to 1.1. In an ideal situation, this ratio should be 1. A ratio varying from 1 is due to the approximation of the expected discrepancy for both techniques (numerical integration and simulation techniques).

In the simulation model, the accuracy of the result depends on the number of simulation. The larger the number of simulation, the higher the accuracy of the result. The accuracy of the approximation in the analytical model is related to the number of nodes chosen in the integral region. To a certain extent, the oscillatory nature of the integral function affects the approximation. In any case, the expected discrepant areas calculated from the numerical integration and from the simulation technique, are close to each other. Both techniques are valid.

The above four examples considered the discrepancy of the spatial features when there is no correlation of nodal errors.

5. CONCLUSIONS

A newly developed analytical model to measure the uncertainty of a spatial feature in 3D GIS was presented in this paper. The uncertainty is determined by the 'true' location and the measured location of the spatial feature. The discrepant area (or volume) was used as a measure of the uncertainty. It was expressed as a mathematical function of which the measured location and the 'true' location of the spatial feature were variables. Given the measured location of the spatial feature, the discrepant area (or volume) could be obtained and the derived result was the discrepant area for this measured location. In general, the measured location may be in the vicinity of the 'true' location. Based on the assumption of error of a spatial feature, a number of possible measured locations were considered in our proposed model rather than one measured location of the spatial feature. Therefore, the expected discrepant area was in the form of a multiple integral. Since this multiple integral could be solved analytically, the Gaussian quadrature, a numerical integration, was implemented to provide an approximate solution for the analytical model. The estimated expected discrepancy was finally compared to the simulated solution.

In our previous uncertainty model for 3D spatial features, the uncertainty model of 3D spatial features was studied using the simulation technique. This simulation model was generated some possible measured locations of a spatial feature based on the same assumption of error of the spatial feature as stated in this paper, and computed the average discrepant area (or volume). In a mathematical point of view, there are a number of infinite points inside a region. However, the simulation model only sampled a certain number of the possible measured location of the spatial feature instead of all possible measured locations of the spatial feature. The accuracy of the expected discrepant area (or volume) is question although more simulations can provide a more precise result. Moreover, the simulation model is quite time-consuming. Thus, we proposed the analytical model by taking all possible measured locations of the spatial feature into account, in order to provide the expected discrepant area (or volume) with great accuracy in real time.

In this paper, an analytical model was provided to validate the simulation model. The numerical results obtained from the analytical model and the simulation results given in our previous study can approximate a similar discrepant area (or volume). However, the calculation for the numerical solution is much faster than that for the simulated solution for the same problem. Therefore, the numerical integration technique is considered the preferred approach in studying the uncertainty of 3D spatial features.

REFERENCE

Burden, R.L. and J.D. Faires. 1993. *Numerical Analysis*, International Thomson Publishing, pp. 211-222.

Burrough, P.A. 1986. *Principles of Geographical Information Systems for Land Resources Assessment*, Oxford: Clarendon Press, pp. 103-135.

Burrough, P.A. and R.A. McDonnell. 1998. *Principles of Geographical Information Systems*, Oxford University Press, pp. 220-240.

Caspary, W. and R. Scheuring. 1992. Error-band as measure of geographic accuracy. *Proceedings of EGIS'92*, pp. 226-233.

Cassettari, S. 1993. *Introduction to Integrated Geo-Information Management*, Chapman & Hall, pp. 2-3.

Dutton, G. 1992. Handling positional uncertainty in spatial databases. *Proceedings of the 5th International Symposium on Spatial Data Handling*, pp. 460-469.

Easa, S.M. 1995. Estimating line segment reliability using Monte Carlo simulation. *Surveying and Land Information System*, vol. 55(3), pp. 136-141.

Heuvelink, G.B.M. 1998. *Error Propagation in Environmental Modeling with GIS*, Taylor and Francis Ltd.

Shi, W.Z. 1994. *Modeling Positional and Thematic Uncertainty in Integration of GIS and Remote Sensing*, ITC Publication, no. 22, Enschede, 148pp.

Shi, W. Z. 1997. Statistical modeling uncertainties of three-dimensional GIS features. *Cartography and Geographic Information Systems*, vol. 24 (1), pp. 21-26.

Shi, W. Z. 1998. A generic statistical approach for modelling error of geometric features in GIS. *International Journal of Geographical Information Science*, vol. 12 (2), pp. 131-143.

Shi, W.Z. and C.K. Cheung. 1999. Simulating reliability of spatial objects in a 3D GIS. *Proceedings of International Symposium on Spatial Data Quality*, July 1999, Hong Kong, pp. 333-342.

Shi, W.Z. and W.B. Liu. 2000. A stochastic process-based model for positional error of line segments in GIS. *International Journal of Geographical Information Science*, vol. 14(1), pp. 51-66.

Stanfel, L. 1996. Reply for discussion of: a generalization on line segment reliability measures. *Surveying and Land Information Systems*, vol. 56(1), pp. 56-57.

Stanfel, L. and C. Stanfel. 1993. A model for the reliability of a line connecting uncertain points. *Surveying and Land Information Systems*, vol. 53(1), pp. 49-52.

Stanfel, L. and C. Stanfel. 1994. A generalization on line segment reliability measures. *Surveying and Land Information Systems*, vol. 54(1), pp. 41-44.